Homework Problems
Physics 451/551
Due October 16, 2018

Solve and submit 2 of these problems on Arnold's material.

1. By expanding out the definitions in terms of the basic one-forms show

$$\omega^1_{\vec{f}} \wedge \omega^1_{\vec{g}} = \omega^2_{\vec{f}\times\vec{g}}$$

$$\omega^1_{\vec{f}} \wedge \omega^2_{\vec{g}} = \left(\vec{f}\cdot\vec{g}\right) dx \wedge dy \wedge dz$$

$$\omega^1_{\vec{f}} \wedge \omega^1_{\vec{g}} = \left(f_x dx + f_y dy + f_z dz\right) \wedge \left(g_x dx + g_y dy + g_z dz\right)$$

$$= \left(f_y g_z - f_z g_y\right) dy \wedge dz + \left(f_z g_x - f_x g_z\right) dz \wedge dx + \left(f_x g_y - f_y g_x\right) dx \wedge dy$$

$$= \omega^2_{\vec{f}\times\vec{g}}$$

$$\omega^1_{\vec{f}} \wedge \omega^2_{\vec{g}} = \left(f_x dx + f_y dy + f_z dz\right) \wedge \left(g_x dy \wedge dz + g_y dz \wedge dx + g_z dx \wedge dy\right) dx \wedge dy \wedge dz$$

$$\left(f_x g_x + f_y g_y + f_z g_z\right) dx \wedge dy \wedge dz = \left(\vec{f}\cdot\vec{g}\right) dx \wedge dy \wedge dz$$

2. Show, using coordinate expressions or otherwise, that the Poisson Bracket solves the Leibniz-like product rule

$$\left[F_1 F_2, F_3\right] = F_1\left[F_2, F_3\right] + F_2\left[F_1, F_3\right]$$

for any three functions F_1, F_2 and F_3.

$$\left[F_1 F_2, F_3\right] = \sum_{i=1}^{n} \frac{\partial\left(F_1 F_2\right)}{\partial q^i}\frac{\partial F_3}{\partial p_i} - \frac{\partial\left(F_1 F_2\right)}{\partial p_i}\frac{\partial F_3}{\partial q^i} =$$

$$\sum_{i=1}^{n} F_1 \frac{\partial\left(F_2\right)}{\partial q^i}\frac{\partial F_3}{\partial p_i} - F_1 \frac{\partial\left(F_2\right)}{\partial p_i}\frac{\partial F_3}{\partial q^i} + \sum_{i=1}^{n} F_2 \frac{\partial\left(F_1\right)}{\partial q^i}\frac{\partial F_3}{\partial p_i} - F_2 \frac{\partial\left(F_1\right)}{\partial p_i}\frac{\partial F_3}{\partial q^i}$$

$$= F_1\left[F_2, F_3\right] + F_2\left[F_1, F_3\right]$$

3. Define the matrix commutator by $\left[M_1, M_2\right] = M_1 M_2 - M_2 M_1$. Show that the commutator operation satisfies the Jacobi identity

$$\left[M_1,\left[M_2, M_3\right]\right] + \left[M_2,\left[M_3, M_1\right]\right] + \left[M_3,\left[M_1, M_2\right]\right] = 0.$$

$$\left[M_1,\left[M_2, M_3\right]\right] = \left[M_1, M_2 M_3 - M_3 M_2\right] = M_1 M_2 M_3 - M_2 M_3 M_1 - M_1 M_3 M_2 + M_3 M_2 M_1$$

Swapping the indices cyclically

$$\left[M_2,\left[M_3, M_1\right]\right] = M_2 M_3 M_1 - M_3 M_1 M_2 - M_2 M_1 M_3 + M_1 M_3 M_2$$

$$\left[M_2,\left[M_3, M_1\right]\right] = M_3 M_1 M_2 - M_1 M_2 M_3 - M_3 M_2 M_1 + M_2 M_1 M_3$$

$$\therefore \left[M_1,\left[M_2, M_3\right]\right] + \left[M_2,\left[M_3, M_1\right]\right] + \left[M_2,\left[M_3, M_1\right]\right] = 0$$

4. In three dimensions let $\omega^1 = xdx + ydy + zdz$.

   a. What is $d\omega^1$?

   $$d\omega^1 = dx \wedge dx + dy \wedge dy + dz \wedge dz = 0$$

   b. What does the Generalized Stoke's Theorem say about the value of

   $$\int_C \omega^1$$

   for any closed curve C?

   Let σ be *any* surface bounded by the closed curve

   $$\int_C \omega = \int_\sigma d\omega = 0.$$

   c. Suppose a curve starts at $(0,0,0)$ and ends at (x, y, z). What is

   $$\int_C \omega^1 ?$$

   (The hard way is to do the line integral. The easy way is to find a function f with $\omega^1 = df$ and use the Generalized Stoke's Theorem.)

   $$\omega^1 = df \qquad f = \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2}$$

   $$\therefore \int_C \omega^1 = \int_C d\,f = f\big|_{x,y,z} - f\big|_{0,0,0} = \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2}$$